ISRAEL JOURNAL OF MATHEMATICS, Vol. 64, No. 3, 1988

INDECOMPOSABILITY OF TREED EQUIVALENCE RELATIONS

BY

SCOT ADAMS

Department of Mathematics, Stanford University, Stanford, CA 94305, USA

ABSTRACT

We define rigorously a "treed" equivalence relation, which, intuitively, is an equivalence relation together with a measurably varying tree structure on each equivalence class. We show, in the nonamenable, ergodic, measure-preserving case, that a treed equivalence relation cannot be stably isomorphic to a direct product of two ergodic equivalence relations.

0. Introduction

Classically, ergodic theory has studied actions of the reals or of the integers on measure spaces. More recently, it has broadened to include actions of Lie groups, algebraic groups and discrete groups. The most recent trend [7, 8, 10, 12] has been to consider any kind of groupoid structure, where the unit space is a measure space. This includes equivalence relations on measure spaces (the case where the groupoid is principal), for which many of the fundamental techniques were developed in [4, 5, 6].

At a conference in Santa Barbara in the late 1970s, Alain Connes suggested the study of equivalence relations with an additional piece of data: a measurably-varying simplicial complex structure on each equivalence class (see [1, Definition 1.7]). In an earlier paper [1, Definition 1.5], we took up a special case of this, where the simplicial complex is a tree (see Definition 1.4 or [1, Definition 1.5]). The present paper continues the study of this special case.

The simplest example of such a "treed" equivalence relation comes from considering the orbits of the action of a finitely-generated free group F acting

Received May 24, 1988

freely on a measure space. A tree structure is then obtained on each orbit by saying that two points x and y are "adjacent" if there is some generator $g \in F$ such that $gx = y$ or $gy = x$. The resulting tree is homogeneous, i.e., has a vertex transitive automorphism group. Thus, in a sense, treed equivalence relations are a groupoid analogue of free groups. In another sense (see [1, Example 1.6.3]), treed equivalence relations are analogous to foliations by manifolds of negative curvature, in the same way as a homogeneous tree is analogous to hyperbolic space (see [9, Chapter II]).

In this paper, we prove an analogue of a theorem of R. Zimmer [14, Theorem 1.4, p. 10] for the category of treed equivalence relations:

THEOREM 4.1. *Let R be a countable, nonamenable, measure-preserving, properly ergodic equivalence relation on a finite measure space M. Assume that (M, R) admits a treeing. Then R is indecomposable, i.e., there do not exist equivalence relations R_1 and R_2 on measure spaces M_1 and M_2 such that R is stably isomorphic to $R_1 \times R_2$.*

Thus, excluding the amenable case (where there is a noticable lack of rigidity, by [15, Theorems 4.3.12 and 4.3.13, p. 82 and 4.3.1(i), p. 84]), any finite measure-preserving treeable equivalence relation is indecomposable. We note that there is overlap between Theorem 4.1 and [14, Theorem 1.1, p. 9]: In [14], if Γ is a free group on two generators (e.g., seen as a lattice in $SL(2, \mathbf{R})$), then the orbit structure on the resulting space S is treeable. So, in this case, either theorem will imply indecomposability.

In §1, we recall the basic definitions regarding equivalence relations and treeings. We also state without proof (but with references) any general results we will need. In §2, we prove the basic lemmas we will need. In §3, we develop a proof of a theorem (Theorem 3.1) which is of independent interest, although its statement is somewhat too complicated to introduce here. It is the main tool in the proof of Theorem 4.1, which appears in §4. Also found in §4 is an application:

COROLLARY 4.3. *Let Γ be a (countable) discrete group. Assume that Γ is a nontrivial amalgam, i.e., that $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$, where Γ_3 is a proper subgroup of Γ_1 and of Γ_2. Let L be an essentially free, properly ergodic Γ-space with finite invariant measure. Then the equivalence relation defined by the orbits of Γ is indecomposable.*

It would be interesting to know if there is an elementary proof of this corollary which does not require the use of treed equivalence relations.

The techniques of this paper can be used to show that a treed equivalence relation admits no strongly normal subrelations, in the sense of [Feldman, Sutherland and Zimmer, *Subrelations of ergodic equivalence relations*, preprint].

The present work covers roughly one quarter of my doctoral dissertation at the University of Chicago under the direction of Robert J. Zimmer, whose guidance has been invaluable in the development of these results.

1. Basic definitions

Some of the exposition in this section is taken *verbatim* from [1, §§1–2].

All measures are assumed σ-additive and σ-finite. All Borel spaces are assumed to be standard. We define $\mathbf{Z}$ to be the integers, $\mathbf{Z}_+$ to be the positive integers and $\mathbf{Z}_+^0$ to be the nonnegative integers. If S is any set, then we denote the number of elements in S by $|S|$.

If R is an equivalence relation on a set X and $x \in X$, then we denote the R-equivalence class of x by $R(x)$. The *R-saturation* of a subset $W \subseteq X$ is the union of all R-equivalence classes that intersect W. A set which is equal to its own R-saturation is said to be *R-invariant*. If $K \subseteq X$ and R is an equivalence relation on X then we denote $R \cap (K \times K)$ by $R \mid K$. If R_1 and R_2 are equivalence relations on sets X_1 and X_2, respectively, then we define the *product equivalence relation* $R_1 \times R_2$ to be the equivalence relation on $X_1 \times X_2$ defined by: $((x_1, x_2), (y_1, y_2)) \in R_1 \times R_2$ iff $(x_1, y_1) \in R_1$ and $(x_2, y_2) \in R_2$.

If X is a locally compact topological space, then we define $C(X)$ to be the Banach space of all continuous complex-valued functions on X vanishing at infinity, under the supremum norm. Let $M(X)$ denote the set of all probability measures on X, viewed as a convex subset of the unit ball in the dual of $C(X)$. If X is compact, then $M(X)$ is compact in the weak-* topology.

If B is a (standard) Borel space, then let B_* denote the set of all subsets of B containing one or two points. We make B_* into a Borel space by indentifying it with $B \times B$ modulo interchange of coordinates.

Let B be a Borel space and let R be an equivalence relation on B whose graph is a Borel subset of $B \times B$ and whose equivalence classes are countable. An *automorphism of* (B, R) is an automorphism $f: B \to B$ of the Borel space B such that $(x, y) \in R \Rightarrow (f(x), f(y)) \in R$. We denote the group of all automorphisms of (B, R) by $\mathrm{Aut}(B, R)$. A measure μ on B is said to be *R-quasi-invariant* if the R-saturation of any μ-null set is again μ-null. A measure μ on B

is said to be *R-invariant* if every automorphism of (B, R) leaves μ invariant (see [4, Corollary 1, p. 294]).

Let M denote the measure space (B, μ). A Borel equivalence relation R on M is said to be *countable* if a.e. equivalence class is countable. If μ is R-quasi-invariant (resp. R-invariant), then we say that R is *measure-preserving* (resp. *quasi-measure-preserving*) on M. Assuming that R is countable and quasi-measure-preserving, then we say that (M, R) is an *equivalence space*.

If R is a Borel equivalence relation on a measure space M, then we say that R is *ergodic* if every R-invariant measurable subset of M is either *null* (measure zero) or *conull* (having complement of measure zero). If some equivalence class of R is conull, then we say that R is *essentially transitive*. If R is ergodic and not essentially transitive, then we say that R is *properly ergodic*.

If (M_1, R_1) and (M_2, R_2) are equivalence spaces, then we say that they are *isomorphic* if there exists a measure class preserving Borel isomorphism from M_1 to M_2 such that for a.e. equivalence class C of R_1, we have that $f(C)$ is an equivalence class of M_2. (Throughout this work, we will say that some property holds "for a.e. equivalence class" if the union of equivalence classes where it holds is a conull set.)

Now assume that (M_1, R_1) and (M_2, R_2) are equivalence spaces with R_1 and R_2 properly ergodic. Then we say that (M_1, R_1) and (M_2, R_2) are *stably isomorphic* if there exist Borel sets P_1 and P_2 of positive measure in M_1 and M_2, respectively, such that $(P_1, R_1 \mid P_1)$ and $(P_2, R_2 \mid P_2)$ are isomorphic.

If (M, R) is an equivalence space, and G is a topological group, then a measurable map $\alpha : R \to G$ is called a *cocycle* if, for a.e. $x \in M$, for all $y, z \in R(x)$, we have $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$.

We now define amenability of an equivalence space as in [11, Definition 3.1, p. 27]: Let E be a separable Banach space and let $\alpha : R \to \mathrm{Iso}(E)$ be a measurable cocycle, where $\mathrm{Iso}(E)$ denotes the topological space of all isometric automorphisms of E with the strong operator topology. (We will denote the action of $\mathrm{Iso}(E)$ on E on the right.) Let $\alpha^* : R \to \mathrm{Iso}(E^*)$ denote the adjoint cocycle, $\alpha^*(x, y) := (\alpha(x, y)^*)^{-1}$. A mapping $x \mapsto A_x$ which associates to each $x \in M$ a weak-* compact, convex subset A_x of the unit ball of E^* is said to be *Borel* if $\{(x, a) \in M \times E^* \mid a \in A_x\}$ is a Borel subset of $M \times E^*$. It is said to be *α^*-invariant* if, for a.e. $x \in M$, for all $y \in R(x)$, $A_x\alpha^*(x, y) = A_y$. We will commonly denote $x \mapsto A_x$ by $(A_x)_{x \in M}$.

Definition 1.1. Let (M, R) be an equivalence space. Let $\alpha : R \to \mathrm{Iso}(E)$ be a measurable cocycle, and let $(A_x)_{x \in M}$ be an α^*-invariant Borel field of weak-* compact, convex subsets of the unit ball of E^*. Then the ordered pair $(\alpha, (A_x)_{x\in M})$ is called an *affine space over* (M, R).

A measurable mapping $f: M \to E^*$ is said to be a *section of* $(A_x)_{x\in M}$ if $f(x)\in A_x$, for a.e. $x\in M$. A section f is said to be α^**-invariant* (or, simply *invariant*) if, for a.e. $x \in M$, for all $y\in R(x)$, we have $f(x)\alpha^*(x, y) = f(y)$.

DEFINITION 1.2. We say that an equivalence space is *amenable* if any affine space over it has an invariant section.

For more information on amenability and its properties, see [15, §4.3, especially pp. 82–84].

All trees are assumed to have countable vertex and edge sets. (For the basic terminology concerning trees, we cite [9].) Let $\mathscr{U}$ denote the tree with infinitely many vertices, and infinitely many edges belonging to each vertex. We call this the *universal tree*. A tree is said to be *locally finite* if every vertex belongs to only finitely many edges. If T is a tree, then we denote the vertex set and (undirected) edge set of T by $V(T)$ and $E(T)$, respectively. If, in addition $v\in V(T)$, then we denote the set of edges in T belonging to v by $E_T(v)$. If $v, w \in V(T)$, then we let $\mathscr{G}_T(v, w)$ denote the ordered sequence of vertices along the geodesic from v to w.

Let T be a tree with distance function $d: V(T)\times V(T)\to \mathbf{Z}^0_+$. If $v\in V(T)$, $n \in \mathbf{Z}^0_+$, then we let $S_T(v, n) := \{w \in V(T) \mid d(v, w) = n\}$. We define the *ends of* T as in [9, Exercise 1, pp. 20–21]. We denote the topological space of ends of T by ∂T. For any $v, w \in V(T)$, we define $\partial_{vw}T$ to be the set of all ends in T whose geodesic to v passes through w. Similarly, we define $V_{vw}(T)$ to be the set of vertices in T whose geodesic to v passes through w.

We define (non-standard) the *trunk vertices of* T to be the collection of vertices of T which lie on a geodesic connecting two ends. The *trunk of* T is the subtree of T whose vertex set consists of the trunk vertices of T.

Next, we head toward the definition of a "treed equivalence space", which is the basic object of study in this work. As a first approximation, we define a "graphed equivalence space":

DEFINITION 1.3. Let (M, R) be an equivalence space. A *graphing of* (M, R) is a symmetric relation $S \subset R$ on B such that S is a Borel subset of $M \times M$. We then say that (M, R, S) is a *graphed equivalence space*.

(Generally, S is neither reflexive nor transitive; it is not an equivalence relation.)

We will abuse notation and use $R(x)$ to refer also to the graph with vertex set $R(x)$ and directed edge set $S \cap [R(x) \times R(x)]$.

The following definition makes rigorous what is meant by "measurably putting a tree structure on each equivalence class" of an equivalence relation.

DEFINITION 1.4. Let (M, R, S) be a graphed equivalence space. If for a.e. $x \in M$, the graph $R(x)$ is a locally finite tree (i.e., it is connected with no circuits, cf. [9, Definition 6, p. 17]), then we say that S is a *treeing of* (M, R) or that (M, R, S) is a *treed equivalence space*.

For standard examples of treed equivalence spaces, we refer to [1, Examples 1.6.1–1.6.4].

If $v \in V(T)$, then we call $|E_T(v)|$ the *valence* of v. If $x \in M$, then we define the *S-valence of* x to be the valence of x as a vertex in the tree $R(x)$.

Let (M, R, S) be a treed equivalence space and let $M_1 \subseteq M$ be a Borel subset. Let $R_1 := R \mid M_1$. We define a subset $S_1 \subseteq S \cap (M_1 \times M_1)$ by letting S_1 be the totality of all $(x, y) \in R_1$ such that the geodesic (in the tree $R(x)$) from x to y does not pass through any vertices inside M_1, except for x and y. We call S_1 the *restriction of* S *to* M_1 and denote S_1 by $S \mid M_1$.

Let (M, R, S) be a treed equivalence space. Let M_0 be the set consisting of those $x \in M$ such that x is a trunk vertex of the tree $R(x)$. Then M_0 is a measurable subset of M and we call $(M_0, R \mid M_0, S \mid M_0)$ the *trunk of* (M, R, S). It is again a treed equivalence space.

Next we restate the definition of a simplification (see [1, Definition 2.1]). The point of §2 of [1] was to prove the existence simplifications. We recall the statement of that theorem as well.

Recall that $\mathscr{U}$ denotes the universal tree. By abuse of notation, we will also use $V(\mathscr{U})$ to denote the measure space of counting measure on the set $V(\mathscr{U})$.

DEFINITION 1.5. Let (M, R, S) be a treed equivalence relation and let $D \subseteq M \times V(\mathscr{U})$ be measurable. Let $\Phi : D \to M$ be a measurable map. Assume, for every $x \in M$, that $D_x := \{\mathscr{V} \in V(\mathscr{U}) \mid (x, \mathscr{V}) \in D\}$ is the set of vertices of some locally finite subtree of $\mathscr{U}$. By abuse of notation, we will denote this subtree by D_x as well. Assume further, for a.e. $x \in M$, that the map $\Phi_x := \mathscr{V} \mapsto \Phi(x, \mathscr{V})$ is a tree isomorphism from D_x to $R(x)$. Then we say that (D, Φ) is a *simplification of* (M, R, S). If there exists $\mathscr{V}_0 \in V(\mathscr{U})$ such that $\Phi(x, \mathscr{V}_0) = x$, for a.e. $x \in M$, then we say that (D, Φ) is *level at* $\mathscr{V}_0$.

Note that, in the following definition, the action of $\mathrm{Aut}(M, R)$ on M is denoted as a *right* action.

DEFINITION 1.6. Let (D, Φ) be a simplification of a treed equivalence space (M, R, S). Let $\mathscr{V}_0 \in V(\mathscr{U})$. Let $\alpha : R \to \mathrm{Aut}(\mathscr{U})$ be a cocycle. Then we say that α is *compatible with* (D, Φ) if, for a.e. $x \in M$, for all $y \in R(x)$, for all $\mathscr{V} \in D_x$, we have $\mathscr{V}\alpha(x, y) \in D_y$ and $\Phi_y(\mathscr{V}\alpha(x, y)) = \Phi_x(\mathscr{V})$.

THEOREM 1.7. *Let (M, R, S) be a treed equivalence space and let $\mathscr{V}_0 \in V(\mathscr{U})$. Then there exists*

(1) *a simplification (D, Φ) of (M, R, S) which is level at $\mathscr{V}_0$ and*

(2) *a cocycle $\alpha : R \to \mathrm{Aut}(\mathscr{U})$ which is compatible with (D, Φ).*

PROOF. See [1, end of §2]. QED

2. Lemmas

In this section we develop the lemmas that will be needed for the proof of the main theorem.

A Borel equivalence relation A on a measure space M is called *recurrent* if, for every $P \subseteq M$ of positive measure, for a.e. $x \in M$, then $R(x) \cap A$ is either empty or infinite. It is well-known that a properly ergodic equivalence relation is recurrent.

In the following, we use the notation $A(x)$ to denote the A-equivalence class of x (cf. §1).

LEMMA 2.1. *Let P be a second countable topological space, A a quasi-invariant recurrent Borel equivalence relation on a measure space M. Let $s : M \to P$ be measurable. Then, for a.e. $x \in M$, there exists a sequence of distinct points $x_1, x_2, \ldots \in A(x) \setminus \{x\}$ such that $s(x_1), s(x_2), \ldots \to s(x)$.*

PROOF. Let $\mathscr{U}$ be a countable base for the topology of P. For $U \in \mathscr{U}$, let Z_U denote the union of all the R-equivalence classes which meet $s^{-1}(U)$ in a finite non-empty set; Z_U is null. Let $Z := \bigcup_{U \in \mathscr{U}} Z_U$. Then $M \setminus Z$ consists of points for which the conclusion holds. QED

The following is well-known, and, in fact does not require G to be discrete. We will be using it, however, so we provide a proof for the reader's convenience.

LEMMA 2.2. *Let M be a measure space and G a (countable) discrete amenable group. Assume that G acts on M and that the action is quasi-measure-preserving. (That is, $G \times M \to M$ is measurable and any element of G carries* *null sets to null sets.) Then the equivalence relation defined by the orbits of G is amenable.*

PROOF. Let R denote the equivalence relation defined by the orbits of G. Let $(\alpha, (A_x)_{x\in M})$ be an affine space over (M, R), where $\alpha: M \to \mathrm{Iso}(E)$, for a separable Banach space E. Let S denote the space of sections over $(\alpha, (A_x)_{x\in M})$ viewed as a weak-* compact, convex subset of the unit ball in $L^\infty(M, E^*) \cong L^1(M, E)^*$. Then G acts on S by $(sg)(x) = s(xg^{-1})\alpha^*(xg^{-1}, x)$. By [15, Proposition 4.1.4, p. 60], we find that G has a fixed point s_0 in S. The section s_0 is easily seen to be α^*-invariant. QED

LEMMA 2.3. *Let (M, R) be an equivalence space. Denote the measure on M by μ. Assume that R is measure-preserving. Let $A, B \subseteq M$ be measurable subsets, and let $f: A \to B$ be a measurable map which respects R, i.e., which satisfies $(x, f(x)) \in R$, for a.e. $x \in A$. Assume that f is essentially surjective, i.e., that $B \setminus f(A)$ is null. Then $\mu(A) \geqq \mu(B)$. If $\mu(A) = \mu(B)$, then f is essentially injective, i.e., $|f^{-1}(\{y\})| = 1$, for a.e. $y \in B$.*

PROOF. By changing A and B by sets of measure zero, we may assume that A and B are Borel. By a measurable selection theorem [2, Theorem 3.4.3, p. 77], there exists a measurable map $s: B \to A$ such that $f(s(y)) = y$, for a.e. $y \in B$. Removing sets of measure zero from A and B, and restricting f and s to the new versions of A and B, we may assume (by standard measure-theoretic techniques) that s is Borel and that $f(s(y)) = y$ for all $y \in B$. The map s is clearly injective and respects R, so, by [4, Corollary 1, p. 294], we have that $\mu(s(B)) = \mu(B)$. But $s(B) \subseteq A$, so $\mu(A) \geqq \mu(B)$.

With the additional assumption that $\mu(B) = \mu(A)$, we conclude that $s(B) = A$ modulo null sets. Since $f \mid s(B)$ is injective, we conclude that $f \mid (A \setminus Z)$ is injective, where Z is the R-saturation of $A \setminus s(B)$. QED

LEMMA 2.4. *Let R be a measure-preserving Borel equivalence relation on a finite measure space M. Assume that R is properly ergodic on M and that a.e. R-equivalence class is countable. Then there exists an equivalence subrelation $A \subseteq R$ such that A is properly ergodic on M and such that A is amenable.*

PROOF. By [4, Theorem 1, p. 291], there exists a countable group G of automorphisms of (M, R), such that the orbits of G are a.e. the equivalence classes of M. By [15, Proposition 9.3.2, p. 170], there exists a countable, properly ergodic, amenable group H of automorphisms of M such that every orbit of H lies in an orbit of G. Let A be the subrelation of R defined by the orbits of H. By Lemma 2.2, A is amenable. QED

LEMMA 2.5. *Suppose S is a treeing of the equivalence space (M, R). For a.e. $x \in M$, assume that the tree $R(x)$ has one or two ends. Then (M, R) is amenable.*

PROOF. This is a special case of [1, Theorem 5.2]. QED

LEMMA 2.6. *Let A and B be amenable equivalence relations on measure spaces L and M. Then the product relation $A \times B$ on $L \times M$ is again amenable.*

PROOF. It is possible to prove this lemma in a straightforward way from the defintions; however, using [3], there is the following trivial proof:

Choose (by [3]) an action of $\mathbf{Z}$ on L such that the orbits are a.e. the equivalence classes of A. Similarly choose a $\mathbf{Z}$-action on M for B. Then the product $(\mathbf{Z} \times \mathbf{Z})$-action on $L \times M$ has the equivalence classes of $A \times B$ as its orbits. By Lemma 2.2, $A \times B$ is amenable. QED

LEMMA 2.7. *In the properly ergodic case, amenability is an invariant of stable isomorphism. That is: Suppose that (L, Q) and (M, R) are equivalence spaces with Q and R properly ergodic. Assume that (L, Q) and (M, R) are stably isomorphic. Then (L, Q) is amenable if and only if (M, R) is amenable.*

PROOF. This is well-known and can be established from the definitions, but we again give a quick proof using advanced tools.

It suffices to show that if (M, R) is a properly ergodic equivalence space and if $P \subseteq M$ is a Borel subset of positive measure, then R is amenable $\Leftrightarrow R \mid P$ is amenable.

$(\Rightarrow)$ Supposing that R is amenable, then, by [3], there exists a $\mathbf{Z}$-action on M whose orbits are the equivalence classes of R. Since $\mathbf{Z}$ is the free group on one generator, this yields a treeing of (M, R), where the a.e. tree is the homogeneous tree with two edges belonging to each vertex. Call this tree the *integer tree* and call the treeing S. Then the restriction $S \mid P$ of S to P (see §1, the remarks following Definition 1.4) is a treeing of $(P, R \mid P)$ in which a.e. tree is the integer tree, hence has two ends. Then, by Lemma 2.5, we are done.

$(\Leftarrow)$ Now suppose that $R \mid P$ is amenable and let $(\alpha, (A_x)_{x \in M})$ be an affine space over (M, R). Upon restriction to P, we can use amenability of $R \mid P$ to find an $(\alpha \mid (R \mid P))^*$-invariant section s over P. By [4, Theorem 1, p. 291], there exists a countable subgroup $G \subseteq \mathrm{Aut}(M, R)$ whose orbits are the equivalence classes of R. Let $g_1, g_2, \ldots$ be a listing of the elements of G. By ergodicity of R it follows that, for a.e. $x \in M$, there exists $g \in G$ such that $xg \in P$; let

$i_x := \min\{i \mid xg_i \in P\}$. Define $s_0(x) := s(xg_{i_x})\alpha^*(xg_{i_x}, x)$. It is straightforward to check that s_0 is an α^*-invariant section. QED

Say that an equivalence space (M, R) is *treeable* if it admits a treeing. It is not *a priori* clear that there are equivalence spaces that are not treeable; in a future paper, we intend to show that an equivalence space with Kazdan's property (T) is not treeable. Further, the present paper shows that a product of two equivalence spaces is not treeable (assuming proper ergodicity and measure-invariance), see Theorem 4.2. However, the following lemma can be used to show that a wide variety of equivalence spaces *are* treeable.

LEMMA 2.8. *In the finite measure-preserving and properly ergodic case, treeability is an invariant of stable isomorphism. That is: Suppose that (L, Q) and (M, R) are stably isomorphic equivalence spaces with Q and R properly ergodic and measure-preserving. Suppose further that the measures on L and on M are both finite. Then there exists a treeing for (L, Q) if and only if there exists a treeing for (M, R).*

PROOF. It is enough to show: Let (M, R) be an equivalence space, and assume that M is a finite measure space and that R is properly ergodic and measure preserving. Let $P \subseteq M$ be a Borel subset of positive measure. Then (M, R) admits a treeing $\Leftrightarrow (P, R, P)$ does.

($\Rightarrow$) Let $S_1 := \{(x, y) \in S \mid x, y \in P\}$. Note that a.e. equivalence class $R(x) \cap P$ becomes a forest under the adjacency defined by S_1. We proceed to connect up the trees in a.e. forest.

For each $z \in M \setminus P$, let A_z denote the set of points in P which are adjacent to z under S. Note that a.e. A_z is finite, so we may choose a linear ordering of a.e. A_z in a way which varies measurably in z. Let S_2 denote the set of $(x, y) \in P \times P$ such that, for some $z \in M \setminus P$, we have that $x, y \in A_z$ and that x and y are consecutive in the ordering on A_z.

For each $z \in M \setminus P$ let B_z denote the set of points in P which are a distance one or two away from z in the tree $R(z)$. Then a.e. B_z is a finite disjoint union of totally ordered finite sets chosen from the A_z's. We extend the total ordering so that any two elements of B_z which were consecutive before remain consecutive. Again, we choose this extended total ordering so that it varies measurably in z. Let S_3 denote the set of $(x, y) \in P \times P$ such that x and y are consecutive in some B_z.

Next, we let C_z denote the points in P which are distance $\leqq 3$ from z. A.e. C_z is a finite disjoint union of totally ordered finite sets chosen from the B_z's. We perform another measurable extension of the total ordering preserving consecutiveness from the previous stage. Let S_4 denote the set of (x, y) such that x and y are consecutive in some C_z.

Continuing in this way, we form an increasing sequence $S_1, S_2, \ldots$. Let $S' := \bigcup_i S_i$. Then it is straightforward to verify that S' gives a treeing of $(P, R \mid P)$.

($\Leftarrow$) Now assume that there exists a treeing S of $(P, R \mid P)$. By [4, Theorem 1, p. 291], there exists a countable subgroup $G \subseteq \mathrm{Aut}(M, R)$ whose orbits are exactly the equivalence classes of R. Let $g_1, g_2, \ldots$ be a listing of the elements of G. Let μ denote the measure on M.

By quasi-invariance, for a.e. $x \in M \setminus P$, there exists $g \in G$ such that $xg \in P$; we define $i_x := \min\{i \mid xg_i \in P\}$. Let $S' := S \cup \{(x, xg_{i_x}), (xg_{i_x}, x) \mid x \in M \setminus P\}$. This is easily seen to define a tree-structure on a.e. $R(x)$, but we are again faced with the problem of showing that this tree structure is locally finite, for a.e. $x \in M$. So assume *for a contradiction* that there exists a Borel set $Q \subseteq P$ of positive measure such that every $x \in Q$ is S'-adjacent to infinitely many elements of $M \setminus P$. Let $M_x := \{y \in M \setminus P \mid (x, y) \in S'\}$, for $x \in Q$. For $x \in Q$, inductively define

$$i_x^1 := \min\{i \mid xg_i \in M_x\},$$

$$i_x^{j+1} := \min\{i \mid xg_i \in M_x, xg_i \neq xg_{i_x^1}, \ldots, xg_i \neq xg_{i_x^j}\} \qquad (j = 1, 2, \ldots).$$

Then $xg_{i_x^1}, xg_{i_x^2}, \ldots$ are the distinct elements of M_x. Let $Q_j := \{xg_{i_x^j} \mid x \in Q\}$. Since R is measure-preserving and since $x \mapsto xg_{i_x^j} : Q \to Q_j$ is essentially injective, essentially surjective and respects R, it follows from [4, Corollary 1, p. 294] that $\mu(Q_j) = \mu(Q)$, for $j = 1, 2, \ldots$. Further, it is easy to see that the sets $Q_1, Q_2, \ldots$. are pairwise disjoint. This clearly *contradicts* the assumption that M has finite measure. QED

Recall from §1 that $\mathcal{U}$ denotes the universal tree and that $\partial\mathcal{U}_*$ denotes the Borel space of all subsets of $\partial\mathcal{U}$ which contain either one or two ends of $\mathcal{U}$.

Let S be a treeing of the equivalence space (M, R). Assume that M is a finite measure space and that R is properly ergodic and measure-preserving. Let (D, Φ) be a simplification of (M, R) which is level at some vertex $\mathcal{V}_0 \in V(\mathcal{U})$ and let $\alpha : R \to \mathrm{Aut}(\mathcal{U})$ be a compatible cocyle. (See Definitions 1.5 and 1.6 and Theorem 1.7.) We will frequently denote a measurable map $x \mapsto F_x : M \to \partial\mathcal{U}_*$ by $(F_x)_{x \in M}$. Note that the action of $\mathrm{Aut}(\mathcal{U})$ on $\partial\mathcal{U}$ induces a (right) action of $\mathrm{Aut}(\mathcal{U})$ on $\partial\mathcal{U}_*$. We say that $(F_x)_{x \in M}$ is *α-invariant* if: for a.e. $x \in M$, for all $y \in R(x)$, we have $F_x \alpha(x, y) = F_y$.

THEOREM 2.9. *Let M, R, S, D, Φ, $\mathscr{V}_0$, α be as above. Assume that there exists an α-invariant system $(F_x)_{x\in M} \subseteq \partial\mathscr{U}$ (see definition above) such that $F_x \subseteq \partial D_x$, for a.e. $x \in M$. Then (M, R) is amenable.*

PROOF. *Case* 1. For x in a set of positive measure, $|F_x| = 2$. Since $x \mapsto |F_x| : M \to \{1, 2\}$ is measurable and is constant on R-equivalence classes, it follows from ergodicity that $|F_x| = 2$, for a.e. $x \in M$. For those $x \in M$, let $C_x \subseteq D_x$ denote the set of all vertices in $\mathscr{U}$ on the geodesic joining the two points in the set F_x. Now let $C \subseteq M$ be defined by

$$C := \{x \in M \mid \mathscr{V}_0 \in C_x\}.$$

Then the α-invariance of $(F_x)_{x\in M}$ implies that $C = \bigcup_{x\in M} \Phi_x(C_x)$ modulo null sets. This shows that C meets a.e. equivalence class, so that C has positive measure, by R-invariance (in fact, by R-quasi-invariance) of the measure on M. Recall (§1, remarks after Definition 1.4) that $S \mid C$ denotes the restriction of S to C. Now $S \mid C$ is a treeing of $(C, R \mid C)$ in which each vertex is adjacent to exactly two other vertices in a.e. $(R \mid C)$-equivalence class. This implies that, under $S \mid C$, a.e. equivalence class of $R \mid C$ becomes a tree with exactly two ends. Then Lemma 2.5 implies that $(C, R \mid C)$ is amenable. And Lemma 2.7 then implies that (M, R) is amenable.

Case 2. For a.e. $x \in M$, $|F_x| = 1$. Let C denote the totality of all $x \in M$ such that the tree $R(x)$ has one or two ends. This is an R-invariant measurable subset of M, so is either null or conull. By Lemma 2.5 it is enough to show that C is conull. So assume *for a contradiction* that C is null, i.e., that a.e. $R(x)$ has $\geqq 3$ ends. Replacing (M, R, S) by its trunk (see §1, the remarks before Definition 1.3), we may assume that a.e. tree $R(x)$ has no pending vertices and that a.e. tree $R(x)$ contains a vertex of valence $\geqq 3$. (A pending vertex is a vertex of valence 1; valence is defined in §1, in the remarks following Definition 1.4. It is an easily proved fact that any tree with $\geqq 3$ ends must contain a vertex of valence $\geqq 3$.) Let X denote the elements of M of S-valence $\geqq 3$. Since X meets a.e. $R(x)$, it follows from R-invariance (indeed, from R-quasi-invariance) of the measure on M that X has positive measure.

For $x \in M$, let $E_x \in \partial\mathscr{U}$ be the unique element of F_x, let $\mathscr{V}_x$ denote first vertex after $\mathscr{V}_0$ on the geodesic from $\mathscr{V}_0$ to E_x and let $f(x) := \Phi_x(\mathscr{V}_x)$. Then $f: M \to M$ is easily seen to be measurable. Since there are no pending vertices, it follows that f is essentially surjective (i.e., that $f(M)$ is conull). In addition, f respects R (i.e., for a.e. $x \in M: (x, f(x)) \in R$). Thus Lemma 2.3 implies that f is essentially injective (i.e., that $|f^{-1}(\{x\})| = 1$, for a.e. $x \in M$).

Let $x \in X$. Then $\mathcal{V}_0 = \Phi_x^{-1}(x)$ has valence $\geqq 3$ in the tree D_x, so there exist three vertices in D_x adjacent to $\mathcal{V}_0$. At least two of them, say v and w, cannot lie on the geodesic from u to E_x. From α-invariance of $(F_x)_{x \in M}$, it follows that $f(\Phi_x(v)) = f(\Phi_x(w)) = x$. So $|f^{-1}(x)| > 1$, for every $x \in X$. As X has positive measure, this *contradicts* essential injectivity of f. QED

3. The two point support theorem

Let A be a recurrent (defined at the start of §2), amenable subrelation of a countable, properly ergodic, measure-preserving equivalence relation R on a finite measure space M. Let S be a treeing of R. As in Definitions 1.5 and 1.6 and Theorem 1.7, let (D, Φ) be a simplification of (M, R, S), level at a vertex $\mathcal{V}_0 \in V(\mathcal{U})$, with compatible cocycle α. We will refer to a measurable map $x \mapsto \mu_x : M \to M(\partial\mathcal{U})$ as a *measures-section* if μ_x is supported in ∂D_x, for a.e. $x \in M$. We will commonly denote $x \mapsto \mu_x$ by $(\mu_x)_{x \in M}$. We say that $(\mu_x)_{x \in M}$ is *α-invariant* if $\mu_x \alpha(x, y) = \mu_y$, for a.e. $x \in M$, for all $y \in R(x)$. (We are, as usual, denoting the natural action of $\mathrm{Aut}(\mathcal{U})$ on $\partial\mathcal{U}$ and on $M(\partial\mathcal{U})$ as a *right* action.)

Recall from §1 that $\partial\mathcal{U}_*$ denotes the Borel space of all one point or two point subsets of the boundary $\partial\mathcal{U}$ of $\mathcal{U}$. There is a natural (right) action of $\mathrm{Aut}(\mathcal{U})$ on $\partial\mathcal{U}_*$.

THEOREM 3.1. *In the above setting, there exists a unique measurable map $x \mapsto F_x : M \to \partial\mathcal{U}_*$ such that:*

(1) *$F_x \subseteq \partial D_x$, for a.e. $x \in M$.*
(2) *If $(\mu_x)_{x \in M}$ is an $\alpha \mid A$-invariant measures-section, then μ_x is supported on F_x, for a.e. $x \in M$; and*
(3) *There exists an $\alpha \mid A$-invariant measures-section $(\mu_x^0)_{x \in M}$ such that the support of μ_x^0* is *F_x, for a.e. $x \in M$.*

Theorem 3.1 and its proof are adapted from a result of Zimmer: [12, Theorem 3.7, p. 50]. Before starting the proof, we will establish two lemmas.

LEMMA 3.2. *Suppose that $a_1, a_2, \ldots \in \mathrm{Aut}(\mathcal{U})$. Fix $\mathcal{V} \in V(\mathcal{U})$ and assume that there exists $F, G \in \partial\mathcal{U}$ such that $\mathcal{V}a_1, \mathcal{V}a_2, \ldots \to F$ and such that $\mathcal{V}a_1^{-1}, \mathcal{V}a_2^{-1}, \ldots \to G$. Then, for all $K \in \partial\mathcal{U} \setminus \{G\}$, $Ka_1, Ka_2, \ldots \to F$.*

Recall from §1 that $\partial_{ww'}\mathcal{U}$ (resp. $V_{ww'}(T)$) means the set of ends (resp. vertices) of T whose geodesic to $\mathcal{W}$ passes through $\mathcal{W}'$.

PROOF. Let U be any neighborhood of F in $\mathscr{U}$; we intend to show that $Ka_i \in U$, for large i.

Let $\mathscr{V} = \mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2, \dots$ be the vertices on the geodesic from $\mathscr{V}$ to F and let $\mathscr{V} = \mathscr{V}_0, \mathscr{V}_{-1}, \mathscr{V}_{-2}, \dots$ be the vertices on the geodesic from $\mathscr{V}$ to G. Since $K \neq G$, it follows that the geodesics from $\mathscr{V}$ to G and to K can only share finitely many vertices; say they share l. Choose m a positive integer such that $\partial v_{m-1} v_m \mathscr{U} \subseteq U$. For sufficiently large i, we will have

(1) $\mathscr{V} a_i \in V_{\mathscr{V}_{m-1}} \mathscr{V}_m(\mathscr{U})$; and

(2) $\mathscr{V} a_i^{-1} \in V_{\mathscr{V}_{-l-m}} \mathscr{V}_{-l-m-1}(\mathscr{U})$.

It suffices to show that, for such i, $Ka_i \in U$.

So fix such an i. Then, by (2), the geodesics from $\mathscr{V} a_i^{-1}$ to $\mathscr{V}$ and to K must share $\mathscr{V}_{-l-m}, \dots, \mathscr{V}_{-1}$ (and possibly more), so they share $\geqq m+1$ vertices. Translating by a_i, we find that the geodesics from $\mathscr{V}$ to $\mathscr{V} a_i$ share $\geqq m+1$ vertices. Thus, by (1), they must share $\mathscr{V}_0, \mathscr{V}_1, \dots, \mathscr{V}_m$. But then $Ka_i \in \partial v_{m-1} v_m \mathscr{U} \subseteq U$. QED

For the next theorem, we fix some notation and definitions. We say that a sequence of automorphisms $a_1, a_2, \dots \in \mathrm{Aut}(\mathscr{U})$ is *unbounded* if, for some (or, equivalently, any) vertex $\mathscr{V} \in V(\mathscr{U})$, we have $d(\mathscr{V}, \mathscr{V} a_i) \to \infty$ as $i \to \infty$, where d denotes the distance function in $\mathscr{U}$. (We are here using a *right* action to denote the natural action of $\mathrm{Aut}(\mathscr{U})$ on the vertices $V(\mathscr{U})$ of $\mathscr{U}$.)

Suppose that $a_1, a_2, \dots \in \mathrm{Aut}(\mathscr{U})$ is such an unbounded sequence and that $\mathscr{V} \in V(\mathscr{U})$. For n a positive integer, and for i sufficiently large (depending on n), the geodesic $\mathscr{G}_u(\mathscr{V}, \mathscr{V} a_i)$ from $\mathscr{V}$ to $\mathscr{V} a_i$ has length $\geqq n$, so it makes sense to refer to the nth vertex on that geodesic. We define the *n-shadow about* $\mathscr{V}$ (with respect to $a_1, a_2, \dots$) to be $\bigcup_{i=1}^{\infty} [S_u(\mathscr{V}, n) \cap \mathscr{G}_u(\mathscr{V}, = \mathscr{V} a_i)]$. (Recall from §1 that $S_u(\mathscr{V}, n)$ denotes the n-sphere about $\mathscr{V}$ in $\mathscr{U}$, i.e., the set of vertices in $\mathscr{U}$ of distance exactly n from $\mathscr{V}$.)

LEMMA 3.3. *Suppose that $a_1, a_2, \dots$ is an unbounded sequence of automorphisms of $\mathscr{U}$. Let $\mu, \nu \in M(\partial\mathscr{U})$ be probability measures with compact support and assume that $\mu a_i \to \nu$ as $i \to \infty$. Then ν is supported on one or two points of ∂T.*

PROOF. Fix a vertex $\mathscr{V} \in V(\mathscr{U})$ and let $\mathscr{V}_i := \mathscr{V} a_i$, for $i = 1, 2, \dots$. Let d denote the distance function between vertices in $\mathscr{U}$.

Note that the theorem is obvious if μ is supported on one point. So we may assume that there exist two distinct vertices $\mathscr{W}_1, \mathscr{W}_2 \in V(\mathscr{U})$ such that $\mu(\partial_{\mathscr{V}\mathscr{W}_k} \mathscr{U}) > 0$ for $k = 1, 2$. Let $m := \max\{d(\mathscr{V}, \mathscr{W}_k) \mid k = 1, 2\}$. Note that

$\mathscr{G}_u(\mathscr{V}, \mathscr{W}_1)$ and $\mathscr{G}_u(\mathscr{V}, \mathscr{W}_2)$ share $< m$ vertices. For $k = 1, 2$, let μ_k denote the restriction of μ to $\partial_{\mathscr{V}\mathscr{W}_k}\mathscr{U}$. Let $\mu_3 := \mu - \mu_1 - \mu_2$. These are not probability measures, but they are finite, compactly supported measures on $\partial\mathscr{U}$.

We *claim*, for all $n = 1, 2, \ldots$, that the n-shadow (see remarks preceding this theorem) about $\mathscr{V}$ is finite. Suppose *for a contradiction* otherwise. Then we may pass to a subsequence and assume that the geodesics from $\mathscr{V}$ to $\mathscr{V}_i$ intersect the n-sphere about $\mathscr{V}$ in a distinct sequence of points $\mathscr{V}_i'$, for $i = 1, 2, \ldots$. Let $i \in \mathbf{Z}_+$ be large enough that $d(\mathscr{V}_i, \mathscr{V}) \geqq m + n$. Since the geodesics $\mathscr{G}_u(\mathscr{V}_i, \mathscr{W}_1 a_i)$ and $\mathscr{G}_u(\mathscr{V}_i, \mathscr{W}_2 a_i)$ share $< m$ vertices and since $d(\mathscr{V}_i, \mathscr{V}) \geqq m + n$, it follows that one of these two geodesics, say the kth, diverges from $\mathscr{G}_u(\mathscr{V}_i, \mathscr{V})$ before reaching $\mathscr{V}_i'$. Then $\mu_k a_i$ is supported on $E_i := \partial_{\mathscr{V}\mathscr{V}_i'}\mathscr{U}$.

Passing to a subsequence again, and possibly switching μ_1 and μ_2, we may assume that $\mu_1 a_i$ is supported on E_i, for $i = 1, 2, \ldots$. It is straightforward to show, for any compact set $E \subseteq \partial\mathscr{U}$, that $E \cap E_i = \varnothing$, for i sufficiently large. This then implies that $\mu_1 a_i \to 0$ weak-*. Then $\nu = \lim_{i\to\infty}(\mu_1 + \mu_2 + \mu_3)a_i$ cannot be a *probability* measure. This is the *contradiction* which *establishes the claim*.

The claim shows that the sequence $\mathscr{V}_1, \mathscr{V}_2, \ldots$ lives in a locally finite subtree of $\mathscr{U}$. By unboundedness, it leaves compact sets, so we may pass to a subsequence and assume that $\mathscr{V}_1, \mathscr{V}_2, \ldots \to F$, for some $F \in \partial\mathscr{U}$. Since $a_1^{-1}, a_2^{-1}, \ldots$ is also unbounded, we may repeat the above argument and assume that $\mathscr{V}a_1^{-1}, \mathscr{V}a_2^{-1}, \ldots \to G$, for some $G \in \partial\mathscr{U}$.

Lemma 3.2 immediately proves (by standard compactness arguments) that if $K \subseteq \partial\mathscr{U} \setminus \{G\}$ is compact and if $U \subseteq \partial\mathscr{U}$ is an open neighborhood of F, then $Ka_i \subseteq U$, for all sufficiently large i.

Next we *claim* that if $\mu(\{G\}) = 0$, then $\nu(\{F\}) = 1$. Given $\varepsilon > 0$ and a neighborhood U of F, it suffices to prove that $\nu(U) > 1 - \varepsilon$. Since $\nu(U) = \lim_{i\to\infty}\mu(Ua_i^{-1})$, it then suffices to show that $\mu(Ua_i^{-1}) > 1 - \varepsilon$, for sufficiently large i. Since μ is compactly supported, there exists a compact set $K \subseteq \partial\mathscr{U} \setminus \{G\}$ such that $\mu(K) > 1 - \varepsilon$. Now apply the statement of the preceding paragraph, *proving the claim*.

So we may assume that $\mu(\{G\}) > 0$. Write $\mu = c_1\mu^1 + c_2\mu^2$, where c_1, c_2 are scalars, where μ^1 is supported on $\{G\}$, where $\mu^2(\{G\}) = 0$ and where μ^1 and μ^2 are both compactly supported probability measures on $\partial\mathscr{U}$. Since $\nu = \lim_{i\to\infty}(c_1\mu^1 + c_2\mu^2)a_i$ is a *probability* measure, it is impossible that $\mu^1 a_i \to 0$ (weak-*) as $i \to \infty$. We thus conclude that it is impossible that $Ga_i \to \infty$. So we may pass to a subsequence and assume that Ga_i converges, say to G'. Then $\mu^1 a_i$ tends toward a point mass at G' and, by the preceding claim, $\mu^2 a_i$ tends toward a point mass at F. Consequently, ν is supported on $\{F, G'\}$, and the lemma is proved. QED

We will not require this, but Lemma 3.3 and its proof are both valid upon replacing $\mathscr{U}$ by any tree. We can now give:

PROOF OF THEOREM 3.1. First, we *claim* that every $\alpha|A$-invariant measures-section is a.e. supported on one or two points. So let $(\mu_x)_{x\in M}$ be αA-invariant. Let $x\in M$ be generic. (That is, all statements made about x in the remainder of this paragraph are claimed to hold for a.e. $x\in M$.) By Lemma 2.1, there exists a sequence $x_1, x_2, \ldots \in A(x)\setminus\{x\}$ of distinct points such that $\mu_{x_i}\to\mu$ as $i\to\infty$. Since the x_i are distinct, it follows that the sequence $\varphi_x^{-1}(x_i)$ leaves compact (i.e., finite) sets in D_x. Since the subtree D_x of $\mathscr{U}$ is locally finite, we see that $\Phi_x^{-1}(x_i)$ eventually leaves any ball in $\mathscr{U}$ about $\mathscr{V}_0$. This and the fact that $\Phi_x^{-1}(x_i) = \mathscr{V}_0\alpha(x_i, x)$ implies that $\alpha(x_1, x), \alpha(x_2, x), \ldots$ is an unbounded sequence of automorphisms of the tree $R(x)$. So $\alpha(x, x_1), \alpha(x, x_2), \ldots$ is also unbounded. Now $\mu_x\alpha(x, x_i) = \mu_{x_i}\to\mu_x$ as $i\to\infty$, so, by Lemma 3.3, μ_x is supported on one or two points. Thus *the claim is proved*.

Now let $\mathscr{B}$ denote the measure algebra of sets modulo null sets in M. Inclusion modulo null sets gives a partial ordering on $\mathscr{B}$. With respect to this ordering, all strict chains are countable (because there are no uncountable, pairwise-disjoint collections of sets of positive measure).

Let $\mathscr{B}_1\subseteq\mathscr{B}$ denote the set of all $B\in\mathscr{B}$ such that, for some $\alpha|A$-invariant measures-section $(\mu_x)_{x\in M}$, $B\subseteq\{x\in M \mid \mathrm{Supp}(\mu_x)| = 2\}$ modulo null sets. In this case, we will call B a *two-set for* $(\mu_x)_{x\in M}$. Note that if $B, C\in\mathscr{B}_1$ are two-sets for $(\mu_x)_{x\in M}, (\nu_x)_{x\in M}$, respectively, then $B\cap C$ is, by the above claim, a two-set for $x\mapsto(1/2)(\mu_x+\nu_x)$, so $\mathscr{B}_1$ is closed under finite union. In fact, by taking weighted averages, $\mathscr{B}_1$ is closed under *countable* union. Since all strict chains are countable, we may apply Zorn's lemma to conclude that $\mathscr{B}_1$ contains a maximal element. In fact, since $\mathscr{B}_1$ is closed under finite union, this maximal element must be unique. Call it B^0.

By definition of $\mathscr{B}_1$, there exists an $\alpha|A$-invariant measures-section, say $(\mu_x^0)_{x\in M}$, such that B^0 is a two-set for $(\mu_x^0)_{x\in M}$. By maximality, $B^0 = \{x\in M \mid |\mathrm{Supp}(\mu_x^0)| = 2\}$, modulo null sets. Now let $F_x := \mathrm{Supp}(\mu_x^0)$, for $x\in M$. Then conclusions (1) and (3) of Theorem 3.1 are immediate.

To prove (2), let $(\mu_x)_{x\in M}$ be an $\alpha|A$-invariant measures-section. For $x\in M$, let $\nu_x := (1/2)(\mu_x+\mu_x^0)$. Let $B := \{x\in M \mid |\mathrm{Supp}(\nu_x)| = 2\}$. By maximality of B^0, we have $B\subseteq B^0$, modulo null sets. Let $x\in M$ be generic. (That is, all statements made about x in the rest of this paragraph are claimed to hold for a.e. $x \in M$.) We wish to show that $\mathrm{Supp}(\mu_x) \subseteq \mathrm{Supp}(\mu_x^0)$, since the latter of these two sets is exactly F_x. In the case where ν_x is supported on one point of $\partial\mathcal{U}$, this point must support both μ_x and μ_x^0, so we are done. By the claim made above, the only other possibility is that ν_x is supported on two points, which we now assume. Then $x \in B \subseteq B^0$, so μ_x^0 is supported on two points. But then $\mathrm{Supp}(\mu_x) \subseteq \mathrm{Supp}(\mu_x^0)$, since, otherwise, the support of $\nu_x = (1/2)(\mu_x + \mu_x^0)$ would include three points. So we are done. QED

4. Proof of the main theorem

In this section, we prove the main theorem of this paper:

THEOREM 4.1. *Let R be a countable, nonamenable, measure-preserving, properly ergodic equivalence relation on a finite measure space M. Assume that (M, R) admits a treeing. Then R is indecomposable, i.e., there do not exist equivalence relations R_1 and R_2 on measure spaces M_1 and M_2 such that R is stably isomorphic to $R_1 \times R_2$.*

PROOF. It is well-known and easy to prove that the existence of an invariant measure is an invariant of stable isomorphism. Thus we see that R_1 and R_2 must be measure-preserving. Next, for $i = 1, 2$, replace M_i by a subset of finite, positive measure, whereupon we may assume that the measures on M_1 and on M_2 are both finite.

Since we are working in the finite measure-preserving, properly ergodic case, we know from Lemmas 2.7 and 2.8 that amenability and treeability are invariants of stable orbit equivalence. Thus Theorem 4.1 follows from Theorem 4.2 below. QED

THEOREM 4.2. *Suppose that L and M are finite measure spaces. Let Q (resp. R) be countable, properly ergodic, measure-preserving equivalence relations on L (resp. M). Assume that $Q \times R$ is nonamenable. Then $(L \times M, Q \times R)$ does not admit a treeing.*

PROOF. Recall from §1 that if B is a Borel space, then B_* denotes the Borel space of all subsets of B containing one or two points. Let $\mathcal{U}$ denote the universal tree. As usual, we denote the actions of $\mathrm{Aut}(\mathcal{U})$ on $V(\mathcal{U})$, on $\partial\mathcal{U}$ and on $\partial\mathcal{U}_*$ all as *right* actions. Let $D_X := \{(x, x) \mid x \in L\}$ be the diagonal relation on L. Similarly, let $D_Y := \{(y, y) \mid y \in M\}$.

Assume *for a contradiction* that there exists a treeing S of $(L \times M, Q \times R)$. Using Theorem 1.7, we may choose a simplification (D, Φ) of

$(L \times M, Q \times R, S)$ which is level at some vertex $\mathscr{V}_0 \in V(\mathscr{U})$ and a cocycle $\alpha : Q \times R \to \mathrm{Aut}(\mathscr{U})$ which is compatible with (D, Φ). By Lemma 2.4, there exist properly ergodic, amenable subrelations $A \subseteq Q$, $B \subseteq R$. Then, by Lemma 2.6, $A \times B$ is amenable. Let $(x, y) \mapsto F_{(x,y)} : L \times M \to \partial \mathscr{V}_*$ be the $\alpha \mid (A \times B)$-invariant map determined by Theorem 3.1. Note that $(x, y) \mapsto F_{(x,y)}$ is then $\alpha \mid (D_X \times B)$- and $\alpha \mid (A \times D_Y)$-invariant. Note further that $D_X \times B$ and $A \times D_Y$ are recurrent subrelations of $Q \times R$.

Let $g \in \mathrm{Aut}(L, Q)$. Then $(x, y) \mapsto F_{(xg^{-1}, y)}\alpha((xg^{-1}, y), (x, y))$ also satisfies the conditions (1)–(3) of Theorem 3.1 (after replacing A in that theorem by the subrelation $D_X \times B$). By the uniqueness assertion in that theorem, it follows that

$$F_{(x,y)} = F_{(g^{-1}x,y)}\alpha((xg^{-1}, y),(x, y)),$$

for a.e. $(x, y) \in L \times M$, for all $g \in \mathrm{Aut}(L, Q)$. Let $G \subseteq \mathrm{Aut}(L, Q)$ be a countable subgroup whose orbits are the equivalence classes of Q (see [4, Theorem 1, p. 291]). For $g \in G$, let $Z_g \subseteq L \times M$ be the set of $(x, y) \in L \times M$ such that the above equation fails. Let $Z := \bigcup_{g \in G} Z_g$; Z is null. Then $((x, y), (x', y)) \in Q \times D_Y$ implies

$$F_{(x,y)} = F_{(x',y)}\alpha((x', y), (x, y)),$$

provided $(x, y) \notin Z$. We conclude that $(x, y) \mapsto F_{(x,y)}$ is $\alpha \mid (Q \times D_Y)$-invariant.

By a similar argument, $(x, y) \mapsto F_{(x,y)}$ is $\alpha \mid (D_X \times R)$-invariant. These two statements together imply that $(x, y) \mapsto F_{(x,y)}$ is α-invariant. By Theorem 2.9, we see that $Q \times R$ is amenable. This *contradicts* an hypothesis of the present theorem. QED

COROLLARY 4.3. *Let Γ be a (countable) discrete group. Assume that Γ is a nontrivial amalgam, i.e., that $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$, where Γ_3 is a proper subgroup of Γ_1 and of Γ_2. Let L be an essentially free, properly ergodic Γ-space with finite invariant measures. Then the equivalence relation defined by the orbits of Γ is indecomposable.*

PROOF. Let R denote the equivalence relation defined by the orbits of Γ. Let M_2 denote the measure space with two points in it, each with measure 1/2. Let $R_2 := M_2 \times M_2$ denote the transitive equivalence relation on M_2.

Let T be the tree associated with the amalgam $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$, as in [9, Theorem 7, p. 32]. This tree has infinitely many ends and admits a Γ-action with a fundamental domain containing exactly two vertices. Using the treed bundle construction of [1, Example 1.6.2], we obtain a treed equivalence space

(M, R, Q) where $M := L \times_\Gamma V(T)$. By this construction, a.e. $R(x)$ is isomorphic to T, which has more than two ends. Also, (M, R) is isomorphic to $(L \times M_2, R \times R_2)$, so M has a finite R-invariant measure. By [1, Theorem 5.1], it then follows that (M, R) is not amenable. Then, by Theorem 4.1, (M, R) is indecomposable. Since (M, R) is isomorphic to $(L \times M_2, R \times R_2)$, it follows that (L, R) is stably isomorphic to (M, R). Indecomposability is, by definition (see Theorem 4.1), an invariant of stable isomorphism, so we see that (L, R) is indecomposable, as asserted. QED

References

1. S. Adams, *Trees and amenable equivalence relations*, Ergodic Theory and Dynamical Systems, to appear.
2. W. Arveson, *An Invitation to C*-algebras*, Springer-Verlag, New York, 1976.
3. A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory and Dynamical Systems **1** (1981), 431–450.
4. J. Feldman and C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras I*, Trans. Am. Math. Soc. **234** (1977), 289–324.
5. J. Feldman and C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras II*, Trans. Am. Math. Soc. **234** (1977), 325–359.
6. J. Feldman, P. Hahn and C. Moore, *Orbit structure and countable sections for actions of continuous groups*, Adv. in Math. **28** (1978), 186–230.
7. G. Mackey, *Ergodic theory and virtual groups*, Math. Ann. **166** (1966), 187–207.
8. A. Ramsay, *Virtual groups and group actions*, Adv. in Math. **6** (1971), 253–322.
9. J.-P. Serre, *Trees*, Springer-Verlag, New York, 1980.
10. R. Zimmer, *Extensions of ergodic group actions*, Ill. J. Math. **20** (1976), 373–409.
11. R. Zimmer, *Hyperfinite factors and amenable ergodic actions*, Inv. Math. **41** (1977), 23–31.
12. R. Zimmer, *Orbit equivalence and rigidity of ergodic actions of Lie groups*, Ergodic Theory and Dynamical Systems **1** (1981), 237–253.
13. R. Zimmer, *Ergodic theory, semisimple Lie groups, and foliations by manifolds of negative curvature*, Publ. Math. IHES **55** (1982), 37–62.
14. R. Zimmer, *Ergodic actions of semisimple groups and product relations*, Ann. Math. **118** (1983), 9–19.
15. R. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.